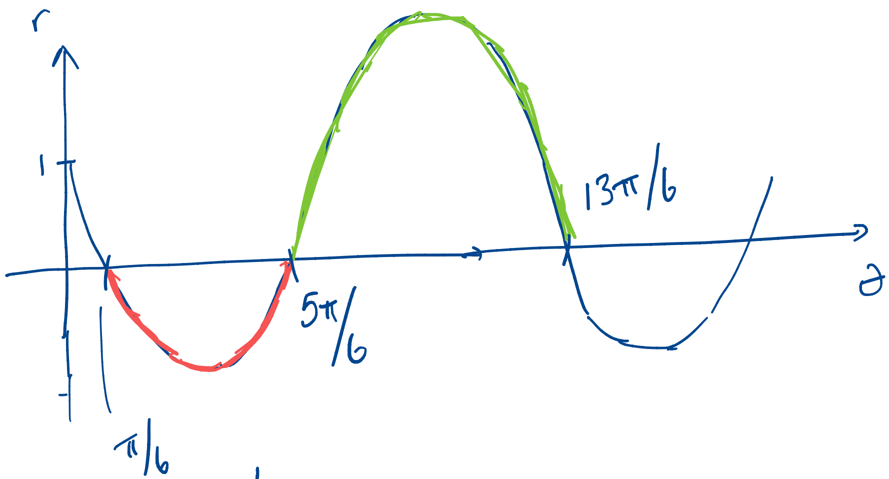
Math 53: Multivariable Calculus **Sections 212, 213**

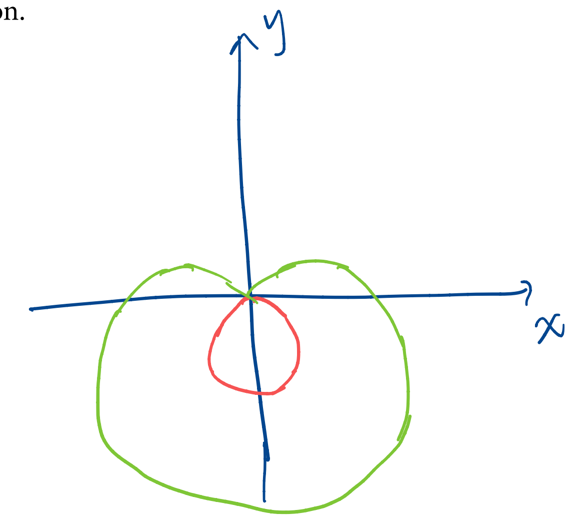
Worksheet for 2020-09-02

Problem 1. Sketch the polar curve $r = 1 - 2\sin\theta$. It is a limaçon with an inner loop.

(a) Set up an integral which computes the arc length of the inner loop (not of the whole curve). The integral is sadly not doable by hand.

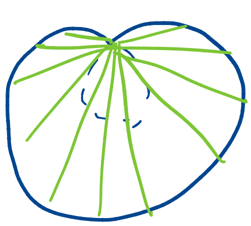
(b) Find the area of the region outside the inner loop but inside of the limaçon.

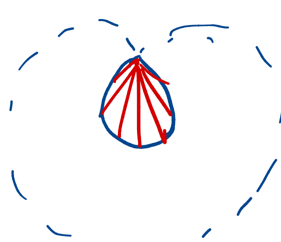




(a) $$\int_{\pi/6}^{5\pi/6} \sqrt{5 - 4\sin\theta}\, d\theta$$

(b) $$\int_{5\pi/6}^{13\pi/6} \frac{1}{2}(1-2\sin\theta)^2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1-2\sin\theta)^2\, d\theta = \boxed{\pi + 3\sqrt{3}}$$





(expand and use cosine double angle formula to rewrite $\sin^2\theta$ as usual to compute the integral)

Problem 2. Consider the portion of the spiral $r = \theta$ with $2\pi/3 \le \theta \le 5\pi/6$. See Figure 1. Compute the area *underneath* this curve in two ways:

(a) Convert to parametric equations and use methods of §10.2.

(b) First compute the area of the region with corners O, B, and D using methods of §10.4. Then use that to find the desired area. **Hint:** Think about the right triangles △BAO and △DCO.

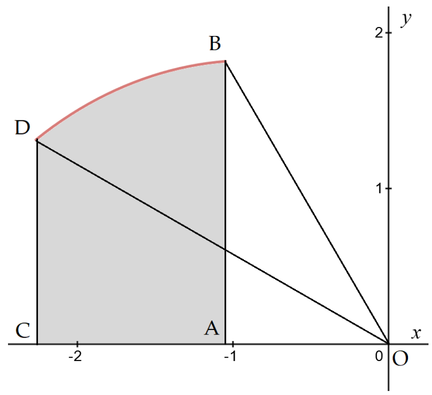


FIGURE 1. The setup of Problem 3.

(a) $x = r\cos\theta = \theta\cos\theta$

$y = r\sin\theta = \theta\sin\theta$

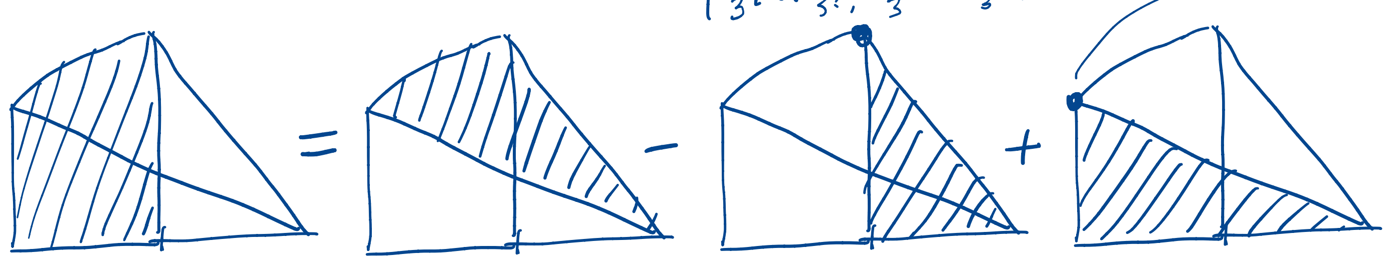
so the shaded area is $\displaystyle\int_{5\pi/6}^{2\pi/3} (\theta\sin\theta)(\cos\theta - \theta\sin\theta)\,d\theta$

After using $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = 1 - 2\sin^2\theta$, and doing some integration by parts, one finds the answer $\boxed{\dfrac{\pi^2\sqrt{3}}{32} + \dfrac{61\pi^3}{1296}}$.

(b) The area of region OBD is $\displaystyle\int_{2\pi/3}^{5\pi/6} \frac{1}{2}\theta^2\,d\theta = \frac{61\pi^3}{1296}$.

$\left(\frac{2\pi}{3}\cos\left(\frac{2\pi}{3}\right), \frac{2\pi}{3}\sin\left(\frac{2\pi}{3}\right)\right)$

$\left(\frac{5\pi}{6}\cos\left(\frac{5\pi}{6}\right), \frac{5\pi}{6}\sin\left(\frac{5\pi}{6}\right)\right)$



$$\boxed{\frac{\pi^2\sqrt{3}}{32} + \frac{61\pi^3}{1296}} = \frac{61\pi^3}{1296} - \frac{\pi^2}{6\sqrt{3}} + \frac{25\pi^2}{96\sqrt{3}}$$

Problem 3. Let $A = (0, 0, 0)$ and $B = (0, 0, 1)$, and let c be a positive real number. Consider the set of all points $P = (x, y, z)$ such that

$$|\overrightarrow{AP}| = c|\overrightarrow{PB}|.$$

Show that when $c = 1$ this set is a plane, and when $c \neq 1$ this set is a sphere.

Write it out: (A bit easier to square the eq. and work w/ that instead)

$$x^2 + y^2 + z^2 = c^2\left(x^2 + y^2 + (z-1)^2\right)$$

$$= c^2x^2 + c^2y^2 + c^2z^2 - c^2 2z + c^2$$

$$(1-c^2)x^2 + (1-c^2)y^2 + (1-c^2)z^2 + c^2 2z - c^2 = 0$$

If $c = 1$ then this simplifies to $z = 1/2$, a plane. Let's suppose $c \neq 1$. Then

$$x^2 + y^2 + z^2 + \frac{2c^2}{1-c^2} z - \frac{c^2}{1-c^2} = 0$$

Complete the square:

$$x^2 + y^2 + \left(z + \frac{c^2}{1-c^2}\right)^2 - \left(\frac{c^2}{1-c^2}\right)^2 - \frac{c^2}{1-c^2} = 0$$

$$x^2 + y^2 + \left(z + \frac{c^2}{1-c^2}\right)^2 = \left(\frac{c^2}{1-c^2}\right)^2 + \frac{c^2}{1-c^2}$$

This is a sphere centered @ $\left(0, 0, -\frac{c^2}{1-c^2}\right)$ with radius $\sqrt{}$